Fast state estimation in linear time-varying systems: an algebraic approach

Yang Tian, Thierry Floquet, Wilfrid Perruquetti

Fast state estimation in linear time-varying systems: an algebraic approach

Yang TIAN, Thierry FLOQUET and Wilfrid PERRUQUETTI

Abstract— **In this note, an algebraic approach for state estimation of linear time-varying (LTV) systems is introduced. This approach is based on the following mathematical tools: Laplace transform, Leibniz formula, operational calculus and distribution theory. A generalized expression of the state variables as a function of the integrals of the output and the input is obtained. The example of a DC motor system and some simulation results are given to illustrate the performance of the proposed approach.**

I. INTRODUCTION

State estimation of linear systems has been extensively studied in the literature because the associated problems are of great interest for engineers. Indeed, the state is not always available by direct measurement and a state observer (a dynamic auxiliary system), which gives a complete estimate based on measurements and inputs, must be designed.

In the context of deterministic linear finite-dimensional time-invariant systems, an observer can be designed if the system is observable, i.e. if any initial state $x(t_0)$ at t_0 can be determined from the knowledge of the system output y and the control u on some time interval $[t_0, t_0+T]$. The observability can be checked by the well-known Kalman rank condition [9] and an observer leading to the asymptotic estimation of the state was first introduced by Luenberger [12].

The observer design problem for linear time-varying systems, that is to say with time-dependent parameters, has also received a particular attention. From [14], Theorem 2.2, if a system is completely observable (the definition will be recalled later), there exists an asymptotically stable observer. Such a type of observers takes the form of Kalman-Bucy filter ([10], [2]) without stochastic terms and some Riccati equation has to be solved. In [16], another time-varying observer was proposed and necessary and sufficient existence conditions were given. Again, the observer matrices have to satisfy a differential equation. In [4], [5], a local "tracking (asymptotic) observer" for flat systems was designed using pole assignment.

The purpose of this article is to design a fast (non asymptotic) reconstructor of the state for LTV systems using an algebraic approach, which is an extension of recent works from M. Fliess and H. S. Ramirez [6], [7] for linear time-invariant systems (see also [13] for signal time derivative estimation or [8] for experimental applications of parameter identification following similar ideas). As a result, the process of estimation is represented by an exact formula, rather than by an auxiliary dynamic system, without any equations to be solved. In this approach, the successive time derivatives of the output are expressed as a function of the integral of the output $y(t)$ itself and of the input $u(t)$ so that the state can be estimated in terms of the integral of $y(t)$ and $u(t)$ in order to attenuate the influence of measurement noise.

The proposed method exhibits the following features:

- the state can be efficiently approximated in a manner that is independent of the initial values,
- computations can be carried out formally by a computer and in a very fast way,
- the observer is robust with respect to additive noise.

The example of a DC motor highlights the efficiency and the robustness properties with respect to noisy measurements of the proposed approach.

II. PROBLEM STATEMENT

Consider the LTV systems given by:

$$\begin{cases} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^d$ is the output. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{d\times n}$ are continuous real matrix functions.

For LTV systems, the definition of completely/totally observable is recalled [11]:

Let $t_f > t_0$. Then, the system (1) is

- completely observable on $[t_0; t_f]$ if any initial state $x(t_0)$ at t_0 can be determined from the knowledge of the output $y(t)$ and the control $u(t)$ on $[t_0; t_f]$;
- totally observable on $[t_0; t_f]$ if it is completely observable on every subinterval of $[t_0; t_f]$.

The observability of the system (1) characterized in terms of $A(t)$, $C(t)$ and their appropriate time derivatives is defined as follows:

Theorem 1: [15] On the interval $[t_0; t_f]$, the system (1) is

- completely observable if rank $O(t) = n$ on $[t_0; t_f]$,
- totally observable if and only if rank $O(t) = n$ on every subinterval of $[t_0; t_f]$,

where $O(t)$ is the observability matrix defined by:

$$\begin{aligned} O(t) &= [S_0(t),\ S_1(t), \ldots,\ S_{n-1}(t)], \\ S_0(t) &= C^T(t), \\ S_{k+1}(t) &= A(t)^T S_k(t) + \dot{S}_k(t), k = 0, \ldots, n-2 \end{aligned} \tag{2}$$

Y. Tian, T. Floquet and W. Perruquetti are with LAGIS (CNRS, UMR 8146), École Centrale de Lille, BP 48, Cité Scientifique, 59650 Villeneuve d'Ascq, France and with Équipe Projet ALIEN, INRIA Lille-Nord Europe. yang.tian@inria.fr, thierry.floquet@ec-lille.fr, wilfrid.perruquetti@ec-lille.fr

In [3], notions about module theory are used to define the observability[1] of the system (1) as the possibility to express all the variables of the system, (in particular all the state variables) as combinations of the components of the input, the output and of their time derivatives up to a finite order.

III. ALGEBRAIC APPROACH AND GENERALIZED EXPRESSION OF STATE ESTIMATION

From now on, only observable monovariable systems are considered, that is to say, $u \in \mathbb{R}$ and $y \in \mathbb{R}$. It is aimed to estimate the state $x(t)$ in a fast way and on the basis of possibly noisy measurements. For this, exact expressions of the state are derived as a function of the integral of the output and the input. Since the integral operator has a filtering effect, the influence of measurement noise can be reduced.

For the sake of convenience, useful formulas are recalled (see [17]):

$$\text{(i)}\quad \mathscr{L}^{-1}\left(\frac{1}{s^l}\frac{d^kY(s)}{ds^k}\right) = \begin{cases} \int \dots \int (-\tau_1)^k y(\tau_1) d\tau_1 \dots d\tau_l, & \text{if } l \geq 1 \\ \frac{d^l\left((-t)^k y(t)\right)}{dt^l}, & \text{if } l \leq 0 \end{cases}$$

$$\text{(ii)}\quad \int \dots \int y(\tau_1) d\tau_1 \dots d\tau_l = \int_0^t \frac{(t-\tau)^{l-1} y(\tau)}{(l-1)!} d\tau$$

$$\text{(iii)}\quad (f * g)(t) = \int_0^t f(t-\lambda) g(\lambda) d\lambda$$

$$\text{(iv)}\quad \int_0^t \delta(\lambda - \lambda_0) f(\lambda) d\lambda = f(\lambda_0)$$

where $\delta(t)$ is a Dirac distribution.

From system (1) which satisfies the (total) observability assumption, one obtains the input-output relation:

$$\sum_{i=0}^{n} a_i(t)\ y^{(i)}(t) = \sum_{i=0}^{m} b_i(t)\ u^{(i)}(t) \tag{3}$$

where $a_n = 1$, $m < n$.

Let us note:

$$\Gamma_0(t) = C(t),$$

$$\Gamma_k(t) = \left(\left(A^T(t) + \frac{d}{dt}\right)^k C^T(t)\right)^T, \quad 0 < k < n$$

$$\Delta_{k0}(t) = \Gamma_k(t) B(t)$$

$$\Delta_{kj}(t) = \begin{cases} C(t)B(t), & \text{if } j = k \\ \Delta_{(k-1)(j-1)}(t) + \frac{d}{dt}\Delta_{(k-1)j}(t), & \text{if } 1 \leq j < k \end{cases}$$

One can show that for all $0 \leq k \leq n-1$:

$$y^{(k)}(t) = \Gamma_k(t) x(t) + \sum_{j=1}^{k} \Delta_{kj}(t) u^{(j-1)}(t) \tag{4}$$

Indeed, one has

$$y(t) = \Gamma_0(t) x$$

$$\dot{y}(t) = \Gamma_1(t) x + \Delta_{11}(t) u$$

[1] which is also valid for nonlinear systems [1], [6].

Assume (4) is true for some integer $k > 0$, one has:

$$\begin{aligned} y^{(k+1)}(t) &= \frac{d}{dt}(\Gamma_k(t))x + \Gamma_k(t)(A(t)x + B(t)u) + \frac{d}{dt}(\Delta_{k1}(t))u \\ &\quad + \Delta_{k1}(t)\dot{u} + \frac{d}{dt}(\Delta_{k2}(t))\dot{u} + \Delta_{k2}(t)u^{(2)} + \frac{d}{dt}(\Delta_{k3}(t))u^{(2)} \\ &\quad + \dots + \Delta_{k(k-1)}(t)u^{(k-1)} + \frac{d}{dt}(\Delta_{kk}(t))u^{(k-1)} + \Delta_{kk}(t)u^{(k)} \\ &= \left(\frac{d}{dt}(\Gamma_k(t)) + \Gamma_k(t)A(t)\right)x + \left(\Delta_{k0}(t) + \frac{d}{dt}(\Delta_{k1}(t))\right)u \\ &\quad + \left(\Delta_{k1}(t) + \frac{d}{dt}(\Delta_{k2}(t))\right)\dot{u} + \left(\Delta_{k2}(t) + \frac{d}{dt}(\Delta_{k3}(t))\right)u^{(2)} \\ &\quad + \dots + \left(\Delta_{k(k-1)}(t) + \frac{d}{dt}(\Delta_{kk}(t))\right)u^{(k-1)} \\ &\quad + \Delta_{(k+1)(k+1)}(t)u^{(k)} \\ &= \Gamma_{k+1}(t)x + \Delta_{(k+1)1}(t)u + \Delta_{(k+1)2}(t)\dot{u} + \Delta_{(k+1)3}(t)u^{(2)} \\ &\quad + \dots + \Delta_{(k+1)k}(t)u^{(k-1)} + \Delta_{(k+1)(k+1)}(t)u^{(k)} \end{aligned}$$

Then it is true for integer $k+1$. Thus, one can express all the state x as a function of y, u and their time derivatives as follows:

$$x(t) = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \vdots \\ \Gamma_{n-1} \end{pmatrix}^{-1} \left[\begin{pmatrix} y \\ \dot{y} \\ y^{(2)} \\ y^{(3)} \\ \vdots \\ y^{(n-1)} \end{pmatrix} - M \begin{pmatrix} u \\ \dot{u} \\ u^{(2)} \\ u^{(3)} \\ \vdots \\ u^{(n-2)} \end{pmatrix}\right] \tag{5}$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \Delta_{11} & 0 & 0 & \dots & 0 \\ \Delta_{21} & \Delta_{22} & 0 & \dots & 0 \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_{(n-1)1} & \Delta_{(n-1)2} & \Delta_{(n-1)3} & \dots & \Delta_{(n-1)(n-1)} \end{pmatrix}$$

The matrix of observability is invertible since the system is observable. Thus, one can recover the state of the system when one has the knowledge of the output and the input and a finite number of their time derivatives. In the following, an algebraic method is developed to obtain a fast and accurate estimate of those variables.

Theorem 2: For the linear time-varying monovariable systems, the estimates of the successive time derivatives of the measured output y are given by:

$$y_e(t) = \frac{1}{(-t)^n}\left(\widetilde{C}_0 - \widetilde{B}_0 - \sum_{j=0}^{n-1} \widetilde{F}_{0,j}\right) \tag{6}$$

$$
\begin{pmatrix} y_e^{(1)}(t) \\ y_e^{(2)}(t) \\ y_e^{(3)}(t) \\ \vdots \\ y_e^{(n-1)}(t) \end{pmatrix} = -\frac{1}{(-t)^n}\tilde{R}\begin{pmatrix} y_e(t) \\ y_e^{(1)}(t) \\ y_e^{(2)}(t) \\ \vdots \\ y_e^{(n-2)}(t) \end{pmatrix}
+\frac{1}{(-t)^n}\left(\begin{pmatrix} \widetilde{C}_1 \\ \widetilde{C}_2 \\ \widetilde{C}_3 \\ \vdots \\ \widetilde{C}_{n-1} \end{pmatrix} - \begin{pmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \\ \widetilde{B}_3 \\ \vdots \\ \widetilde{B}_{n-1} \end{pmatrix} - \sum_{j=0}^{n-p-1}\begin{pmatrix} \widetilde{F}_{1,j} \\ \widetilde{F}_{2,j} \\ \widetilde{F}_{3,j} \\ \vdots \\ \widetilde{F}_{n-1,j} \end{pmatrix}\right) \tag{7}
$$

with

$$
\tilde{R} = \begin{pmatrix} \alpha_{1,1} & 0 & 0 & \dots & 0 \\ \alpha_{2,1} & \alpha_{2,2} & 0 & \dots & 0 \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \alpha_{n-1,3} & \dots & \alpha_{n-1,n-1} \end{pmatrix}
$$

$$
\alpha_{p,l} = \sum_{j=n-p}^{n-1}\gamma_j \tilde{r}_{l-1,w} + \tilde{d}_{p,l-1}
$$

$$
\gamma_j = \frac{n!n!}{j!j!(n-j)!}, \quad w = p+j-n
$$

$$
\tilde{r}_{g,w} = \binom{w}{g}\frac{j!(-1)^j t^{j-w+g}}{(j-w+g)!}, \quad \tilde{d}_{p,k} = \binom{p}{k}\frac{n!(-1)^n t^{n-p+k}}{(n-p+k)!}
$$

$$
\widetilde{B}_p = \frac{1}{(n-p-1)!}\sum_{i=0}^{n-1}\left((-1)^i\int_0^t \widetilde{E}_{p,a_i,i}y(\lambda)d\lambda\right)
$$

$$
\widetilde{C}_p = \frac{1}{(n-p-1)!}\sum_{i=0}^{m}\left((-1)^i\int_0^t \widetilde{E}_{p,b_i,i}u(\lambda)d\lambda\right)
$$

$$
\widetilde{E}_{p,f_i,i} = \{(t-\lambda)^{n-p-1}(-\lambda)^n f_i(\lambda)\}^{(i)}
$$

$$
\widetilde{F}_{p,j} = \gamma_j\int_0^t \frac{(t-\tau)^{-w-1}(-\tau)^j y(\tau)}{(-w-1)!}d\tau
$$

Proof

a) Apply the Laplace transform to the I/O relation (3)

$$
s^n y(s) - \dots - y^{(n-1)}(0) + \sum_{i=0}^{n-1}\mathscr{L}\left(a_i(t)\ y^{(i)}(t)\right) = \sum_{i=0}^{m}\mathscr{L}\left(b_i(t)\ u^{(i)}(t)\right).
$$

b) Algebraic manipulations.

Deriving the preceding expression n times with respect to s, in order to eliminate the initial conditions, using the Leibniz formula

$$
\frac{d^h(x(s)y(s))}{ds^h} = \sum_{j=0}^{h}\binom{h}{j}\frac{d^{h-j}(x(s))}{ds^{h-j}}\frac{d^j(y(s))}{ds^j}
$$

and the relation

$$
\frac{d^k(s^l)}{ds^k} = \begin{cases} \frac{l!}{(l-k)!}s^{l-k}, & \text{if } 0<k\le l \\ 0, & \text{if } 0<l<k \\ \frac{(-1)^k(k-l-1)!}{(-l-1)!}s^{l-k}, & \text{if } l<0<k \end{cases} \tag{8}
$$

Setting $\gamma_j = \frac{n!n!}{j!j!(n-j)!}$, one gets:

$$
\sum_{j=0}^{n}\gamma_j s^j\frac{d^j(y(s))}{ds^j} + \sum_{i=0}^{n-1}\frac{d^n\mathscr{L}\left(a_i(t)\ y^{(i)}(t)\right)}{ds^n} = \sum_{i=0}^{m}\frac{d^n\mathscr{L}\left(b_i(t)\ u^{(i)}(t)\right)}{ds^n}.
$$

Multiply each side of this expression by $s^{-(n-p)}$:

$$
\sum_{j=0}^{n}\gamma_j\frac{s^j}{s^{n-p}}\frac{d^j(y(s))}{ds^j} + \sum_{i=0}^{n-1}\frac{1}{s^{n-p}}\frac{d^n\mathscr{L}\left(a_i(t)\ y^{(i)}(t)\right)}{ds^n} = \sum_{i=0}^{m}\frac{1}{s^{n-p}}\frac{d^n\mathscr{L}\left(b_i(t)\ u^{(i)}(t)\right)}{ds^n}. \tag{9}
$$

c) Return to time domain.
Applying the inverse Laplace transform to (9), one gets:

$$
\underbrace{\sum_{j=0}^{n}\mathscr{L}^{-1}\left(\frac{\gamma_j}{s^{n-p-j}}\frac{d^j(y(s))}{ds^j}\right)}_{\widetilde{A}_p} + \underbrace{\sum_{i=0}^{n-1}\mathscr{L}^{-1}\left(\frac{1}{s^{n-p}}\frac{d^n\mathscr{L}\left(a_i(t)\ y^{(i)}(t)\right)}{ds^n}\right)}_{\widetilde{B}_p}
= \underbrace{\sum_{i=0}^{m}\mathscr{L}^{-1}\left(\frac{1}{s^{n-p}}\frac{d^n\mathscr{L}\left(b_i(t)\ u^{(i)}(t)\right)}{ds^n}\right)}_{\widetilde{C}_p}. \tag{10}
$$

Now, one needs to express $\widetilde{A}_p$, $\widetilde{B}_p$ et $\widetilde{C}_p$ as a function of y, u and their successive derivatives of order less than p.
c1) Using the two formulas (i) and (ii), one gets:

$$
\mathscr{L}^{-1}\left(\frac{1}{s^l}\frac{d^kY(s)}{ds^k}\right) = \int_0^t \frac{(t-\tau)^{l-1}(-\tau)^k y(\tau)}{(l-1)!}d\tau, \quad l\ge 1
$$

Then,

$$
\mathscr{L}^{-1}\left(\frac{1}{s^{n-p-j}}\frac{d^j(y(s))}{ds^j}\right) =
\begin{cases} \int_0^t \frac{(t-\tau)^{n-p-j-1}(-\tau)^j y(\tau)}{(n-p-j-1)!}d\tau, & \text{if } 0\le j\le n-p-1 \\ \frac{d^{p+j-n}\left((-t)^j y(t)\right)}{dt^{p+j-n}}, & \text{if } n-p\le j\le n \end{cases}. \tag{11}
$$

Setting $w = p+j-n$ and applying the Leibniz formula and the relation (8), one has:

$$
\frac{d^w\left((-t)^j y(t)\right)}{dt^w} = \sum_{g=0}^{w}\binom{w}{g}\frac{j!(-1)^j t^{j-w+g}y^{(g)}(t)}{(j-w+g)!}. \tag{12}
$$

When $j=n$, one has:

$$
\frac{d^p\left((-t)^n y(t)\right)}{dt^p} = (-t)^n y^{(p)}(t) + \sum_{k=0}^{p-1}\binom{p}{k}\frac{n!(-1)^n t^{n-p+k}y^{(k)}(t)}{(n-p+k)!}. \tag{13}
$$

Using (11), (12) and (13), $\widetilde{A}_p$ can be rewritten as follows:

$$
\widetilde{A}_p = \sum_{j=0}^{n-p-1}\widetilde{F}_{p,j} + \sum_{j=n-p}^{n-1}\gamma_j\sum_{g=0}^{w}\binom{w}{g}\frac{j!(-1)^j t^{j-w+g}y^{(g)}(t)}{(j-w+g)!}
+(-t)^n y^{(p)}(t) + \sum_{k=0}^{p-1}\binom{p}{k}\frac{n!(-1)^n t^{n-p+k}y^{(k)}(t)}{(n-p+k)!}, \tag{14}
$$

where

$$\widetilde{F}_{p,j} = \gamma_j \int_0^t \frac{(t-\tau)^{-w-1}(-\tau)^j y(\tau)}{(-w-1)!} d\tau.$$

c2) In order to express $\widetilde{B}_p$ and $\widetilde{C}_p$, one applies the convolution theorem given by:

$$\mathscr{L}^{-1}\left(g_1(s)g_2(s)\right) = g_1(t) * g_2(t).$$

This leads to

$$\widetilde{B}_p = \sum_{i=0}^{n-1} \frac{t^{n-p-1}\varepsilon(t)}{(n-p-1)!} * (-t)^n a_i(t) y^{(i)}(t),$$

where $\varepsilon(t)$ is the step function.

If g_1 is a C^1–function such that $g_1(0) = 0$ and g_2 is a C^0–function then $\int_0^t g_1(t-\lambda)g_2(\lambda)d\lambda = \left[g_1(t-\lambda)\int_0^\lambda g_2(\mu)d\mu\right]_0^t - \int_0^t \frac{dg_1(t-\lambda)}{d\lambda}\left(\int_0^\lambda g_2(\mu)d\mu\right)d\lambda = \int_0^t \frac{dg_1(t-\lambda)}{d(t-\lambda)}\left(\int_0^\lambda g_2(\mu)d\mu\right)d\lambda$. This result can be extended for two distributions g_1 and g_2 with left hand side limited supports. This implies the existence of the convolution product $g_1 * g_2$ and leads to the following more general result

$$\int_0^t g_1'(t-\lambda)g_2(\lambda)d\lambda = \int_0^t g_1(t-\lambda)g_2'(\lambda)d\lambda$$

which reads as

$$g_1'(t) * g_2(t) = g_1(t) * g_2'(t), \tag{15}$$

where the prime notation denotes the distribution derivation.

Using the formulas which were indicated at the beginning and (15), one has:

$$\begin{aligned}
& t^{n-p-1}\varepsilon(t) * (-t)^n a_i(t) y^{(i)}(t) \\
& \overset{(15)}{=} \left[(n-p-1)t^{n-p-2}\varepsilon(t) + t^{n-p-1}\delta(t)\right] * \int_0^t (-\tau_1)^n a_i y^{(i)} d\tau_1 \\
& \overset{(iii)(iv)}{=} (n-p-1)t^{n-p-2}\varepsilon(t) * \int_0^t (-\tau_1)^n a_i y^{(i)} d\tau_1 \\
& \overset{(15)}{=} (n-p-1)!\varepsilon(t) * \int_0^{(n-p-1)} (-t)^n a_i y^{(i)} \\
& \overset{(iii)}{=} (n-p-1)! \int_0^{(n-p)} \varepsilon(t-\tau_1)(-\tau_1)^n a_i y^{(i)} \\
& = (n-p-1)! \int_0^{(n-p)} (-t)^n a_i y^{(i)} \\
& \overset{(ii)}{=} \int_0^t (t-\tau_1)^{n-p-1}(-\tau_1)^n a_i y^{(i)} d\tau_1
\end{aligned}$$

where the following notations were used:

$$\int_0^{(k)} (-t)^n \phi = \int_0^t \ldots \int_0^{\tau_2} (-\tau_1)^n \phi(\tau_1) d\tau_1 \ldots d\tau_k,$$

$$\begin{aligned}
& \int_0^{(k)} \varepsilon(t-\tau_1)(-\tau_1)^n \phi \\
& \quad = \int_0^t \ldots \int_0^{\tau_2} \varepsilon(\tau_2-\tau_1)(-\tau_1)^n \phi(\tau_1) d\tau_1 \ldots d\tau_k.
\end{aligned}$$

So

$$\widetilde{B}_p = \sum_{i=0}^{n-1} \int_0^t \frac{(t-\lambda)^{n-p-1}}{(n-p-1)!}(-\lambda)^n a_i(\lambda) y^{(i)}(\lambda) d\lambda.$$

Then, applying the integration by parts, which can be generalized for the function of class C^i:

$$\begin{aligned}
& \int_a^b f(\lambda) g^{(i)}(\lambda) d\lambda \\
& = \left[\sum_{k=0}^{i-1} (-1)^k f^{(k)}(\lambda) g^{(i-1-k)}(\lambda)\right]_a^b + (-1)^i \int_a^b f^{(i)}(\lambda) g(\lambda) d\lambda
\end{aligned}$$

one gets:

$$\begin{aligned}
\widetilde{B}_p = & \frac{1}{(n-p-1)!} \sum_{i=0}^{n-1} \left[\sum_{j=0}^{i-1} (-1)^j \widetilde{E}_{p,a_i,j} y^{(i-j-1)}(\lambda)\right]_0^t \\
& + \frac{1}{(n-p-1)!} \sum_{i=0}^{n-1} (-1)^i \int_0^t \widetilde{E}_{p,a_i,i} y(\lambda) d\lambda,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{E}_{p,a_i,j} & = \{(t-\lambda)^{n-p-1}(-\lambda)^n a_i(\lambda)\}^{(j)} \\
& = \{(\lambda^2 - t\lambda)^{n-p-1}(-\lambda)^{p+1} a_i(\lambda)\}^{(j)} \\
& = \sum_{f=0}^{j} \binom{j}{f} \frac{d^{j-f}\{(\lambda^2-t\lambda)^{n-p-1}\}}{d\lambda^{j-f}} \frac{d^f\{(-\lambda)^{p+1}a_i(\lambda)\}}{d\lambda^f} \\
& = \sum_{f=0}^{j} \binom{j}{f} \frac{d^{j-f}((\lambda^2-t\lambda)^{n-p-1})}{d(\lambda^2-t\lambda)^{j-f}} \frac{d(\lambda^2-t\lambda)^{j-f}}{d\lambda^{j-f}} \frac{d^f\{(-\lambda)^{p+1}a_i(\lambda)\}}{d\lambda^f}.
\end{aligned}$$

Using the relation (8), one gets

$$\begin{aligned}
& \frac{d^{j-f}((\lambda^2-t\lambda)^{n-p-1})}{d(\lambda^2-t\lambda)^{j-f}} \\
& = \begin{cases} \frac{(n-p-1)!(\lambda^2-t\lambda)^{n-p-1-j+f}}{(n-p-1-j+f)!}, & \text{if } j-f \leq n-p-1 \\ 0, & \text{if } n-p-1 < j-f \end{cases}
\end{aligned}$$

So

$$\sum_{j=0}^{i-1} (-1)^j \left[\widetilde{E}_{p,a_i,j} y^{(i-j-1)}(\lambda)\right]_0^t = 0,$$

$$\widetilde{B}_p = \frac{1}{(n-p-1)!} \sum_{i=0}^{n-1} \left((-1)^i \int_0^t \widetilde{E}_{p,a_i,i} y(\lambda) d\lambda\right). \tag{16}$$

Applying the same operation for $\widetilde{C}$, one gets:

$$\widetilde{C}_p = \frac{1}{(n-p-1)!} \sum_{i=0}^{m} \left((-1)^i \int_0^t \widetilde{E}_{p,b_i,i} u(\lambda) d\lambda\right). \tag{17}$$

Substituting the preceding results (14), (16) and (17) into (10), one obtains the following expressions for the time derivatives of y:

$$y^{(p)}(t) = \frac{1}{(-t)^n}\left(\widetilde{C}_p - \widetilde{B}_p - \sum_{j=0}^{n-p-1} \widetilde{F}_{p,j} - \Gamma_p\right) \tag{18}$$

with

$$\begin{aligned}
\Gamma_p = & \sum_{j=n-p}^{n-1} \gamma_j \sum_{g=0}^{w} \binom{w}{g} \frac{j!(-1)^j t^{j-w+g} y^{(g)}(t)}{(j-w+g)!} + \sum_{k=0}^{p-1} \binom{p}{k} \frac{n!(-1)^n t^{n-p+k} y^{(k)}(t)}{(n-p+k)!} \\
= & (\gamma_{n-p}\tilde{r}_{0,0} + \gamma_{n-p+1}\tilde{r}_{0,1} + \ldots + \gamma_{n-1}\tilde{r}_{0,p-1}) + \tilde{d}_{p,0} + \ldots \\
& + (\gamma_{n-p}\tilde{r}_{p-1,0} + \gamma_{n-p+1}\tilde{r}_{p-1,1} + \ldots + \gamma_{n-1}\tilde{r}_{p-1,p-1}) + \tilde{d}_{p,p-1} \\
= & \sum_{j=n-p}^{n-1} \gamma_j \tilde{r}_{0,w} + \tilde{d}_{p,0} + \ldots + \sum_{j=n-p}^{n-1} \gamma_j \tilde{r}_{p-1,w} + \tilde{d}_{p,p-1} \\
= & \sum_{l=1}^{p} \alpha_{p,l}
\end{aligned}$$

In the particular case where $p=0$, applying the result of (14), (16) and (17), one gets:

$$\widetilde{A}_0 = (-t)^n y(t) + \sum_{j=0}^{n-1} \gamma_j \frac{\int_0^t (t-\tau)^{n-j-1}(-\tau)^j y(\tau) d\tau}{(n-j-1)!}$$

$$\widetilde{B}_0 = \sum_{i=0}^{n-1} \frac{(-1)^i \int_0^t \{(t-\lambda)^{n-1}(-\lambda)^n a_i(\lambda)\}^{(i)} y(\lambda) d\lambda}{(n-1)!}$$

$$\widetilde{C}_0 = \sum_{i=0}^{m} \frac{(-1)^i \int_0^t \{(t-\lambda)^{n-1}(-\lambda)^n b_i(\lambda)\}^{(i)} u(\lambda) d\lambda}{(n-1)!}$$

Clearly, $y_e(t)$ can be rewritten as in (6) as a function of the integral of the output y and the input u. Then, one substitutes $y_e(t)$ in (18) such that one obtains (7) as an expression of the estimate of the successive time derivatives of the measured output y. Due to the triangular structure of the matrix $\tilde{R}$, one gets the estimate of the $p-th$ time derivative of y as a function of the integral of y and the input u only.

State reconstructor

Using relation (5), one obtains the estimate of the state:

$$x_e(t) = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_{n-1} \end{pmatrix}^{-1} \left[\begin{pmatrix} y_e \\ \dot{y}_e \\ \vdots \\ y_e^{(n-1)} \end{pmatrix} - M \begin{pmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-2)} \end{pmatrix} \right].$$

Note that all these computations are singular at $t=0$ but becomes valid for any arbitrary small instant. Therefore one must to evaluate the formula not at $t=0$ but after a small time ε.

IV. EXAMPLE

Consider a simplified model of a DC motor system (electric part is neglected), given by

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{1}{\tau(t)} x_2(t) + \frac{k}{\tau(t)} u(t) \end{cases}$$

with $y = x_1$ as measured output; x_1 is the angular position of the rotor, x_2 is the angular velocity of the rotor and u is the control input voltage. k is strictly positive constant and $\tau(t)$ is time-varying strictly positive parameter.

A. Algebraic approach

Write the I/O relation:

$$y^{(2)}(t) + \frac{1}{\tau(t)}\dot{y} = \frac{k}{\tau(t)} u(t) \tag{19}$$

Step 1: Express $y^{(1)}$ as a function of y, u and their integral.

a) Apply the Laplace transform to the relation (19).

$$s^2 y(s) - s y(0) - \dot{y}(0) + \mathscr{L}\left(\frac{\dot{y}}{\tau(t)}\right) = k\mathscr{L}\left(\frac{u(t)}{\tau(t)}\right) \tag{20}$$

b) Algebraic manipulations.

Derive (20) twice to eliminate the initial conditions:

$$2y(s) + 4s\frac{dy(s)}{ds} + s^2\frac{d^2y(s)}{ds^2} + \frac{d^2\mathscr{L}\left(\frac{\dot{y}}{\tau(t)}\right)}{ds^2} = k\frac{d^2\mathscr{L}\left(\frac{u(t)}{\tau(t)}\right)}{ds^2} \tag{21}$$

Multiply each side of (21) by s^{-1}:

$$\frac{2}{s}y(s) + 4\frac{dy(s)}{ds} + s\frac{d^2y(s)}{ds^2} + \frac{1}{s}\frac{d^2\mathscr{L}\left(\frac{\dot{y}}{\tau(t)}\right)}{ds^2} = \frac{k}{s}\frac{d^2\mathscr{L}\left(\frac{u(t)}{\tau(t)}\right)}{ds^2} \tag{22}$$

c) Return to time domain.

Apply the inverse Laplace transform to (22) using the expressions of (14), (16), (17) and one gets:

$$\widetilde{A}_1 = \int_0^t 2y(\lambda)d\lambda - 2ty(t) + t^2\dot{y}$$

$$\widetilde{B}_1 = \int_0^t \lambda^2\left(\frac{\dot{y}(\lambda)}{\tau(\lambda)}\right)d\lambda = \frac{t^2 y(t)}{\tau(t)} - \int_0^t y(\lambda) d\frac{\lambda^2}{\tau(\lambda)}$$

$$\widetilde{C}_1 = \mathscr{L}^{-1}\left(\frac{k}{s}\frac{d^2\mathscr{L}\left(\frac{u(t)}{\tau(t)}\right)}{ds^2}\right) = k\int_0^t \lambda^2\frac{u(\lambda)}{\tau(t)}d\lambda$$

Using the expression (7) or the relation (10), one obtains:

c1) $\tau(t) = a_0 t + a_1$

$$y^{(1)} = \frac{k\int_0^t \frac{\lambda^2 u(\lambda)}{a_0\lambda + a_1}d\lambda + \int_0^t y(\lambda)\frac{a_0\lambda^2 + 2a_1\lambda}{(a_0\lambda+a_1)^2}d\lambda - \frac{t^2 y(t)}{\tau(t)} - \int_0^t 2y(\lambda)d\lambda + 2ty(t)}{t^2} \tag{23}$$

c2) $\tau(t) = b_0 \sin b_1 t + b_2$

$$y^{(1)} = \frac{\int_0^t y(\lambda)\left(\frac{2\lambda}{b_0\sin(b_1\lambda)+b_2} - \frac{\lambda^2 b_0 b_1\cos(b_1\lambda)}{(b_0\sin(b_1\lambda)+b_2)^2} - 2\right)d\lambda}{t^2} + \frac{\int_0^t \frac{k\lambda^2 u(\lambda)}{b_0\sin(b_1\lambda)+b_2}d\lambda + 2ty(t) - \frac{t^2 y(t)}{b_0\sin(b_1 t)+b_2}}{t^2} \tag{24}$$

Step 2: Express y_e as a function of the integral of y.

Multiply each side of (21) by s^{-2}:

$$\frac{2}{s^2}y(s) + \frac{4}{s}\frac{dy(s)}{ds} + \frac{d^2y(s)}{ds^2} + \frac{1}{s^2}\frac{d^2\mathscr{L}\left(\frac{\dot{y}}{\tau(t)}\right)}{ds^2} = \frac{k}{s^2}\frac{d^2\mathscr{L}\left(\frac{u(t)}{\tau(t)}\right)}{ds^2} \tag{25}$$

Apply the inverse Laplace transform to (25) using the expressions of (14), (16), (17) with $p=0$ and one gets:

$$\widetilde{A}_0 = \int_0^t\int_0^\phi 2y(\lambda)d\lambda d\phi - \int_0^t 4\lambda y(\lambda)d\lambda + t^2 y(t)$$

$$\widetilde{B}_0 = \int_0^t (t-\lambda)\lambda^2\frac{\dot{y}(\lambda)}{\tau(\lambda)}d\lambda = \int_0^t y(\lambda)d\frac{\lambda^3}{\tau(\lambda)} - t\int_0^t y(\lambda)d\frac{\lambda^2}{\tau(\lambda)}$$

$$\widetilde{C}_0 = \mathscr{L}^{-1}\left(\frac{k}{s^2}\frac{d^2\mathscr{L}\left(\frac{u(t)}{\tau(t)}\right)}{ds^2}\right) = k\int_0^t (t-\lambda)\lambda^2\frac{u(\lambda)}{\tau(t)}d\lambda$$

Using the expression (6) or the relation (10), one obtains:

c1) $\tau(t) = a_0 t + a_1$

$$y_e(t) = \frac{\int_0^t y(\lambda)\left(\frac{-\left(2a_0\lambda^3 + 3a_1\lambda^2 - t(a_0\lambda^2 + 2a_1\lambda)\right)}{(a_0\lambda+a_1)^2} + 6\lambda - 2t\right)d\lambda}{t^2} + k\frac{\int_0^t \frac{(t-\lambda)\lambda^2 u(\lambda)}{a_0\lambda+a_1}d\lambda}{t^2} \tag{26}$$

c2) $\tau(t) = b_0 \sin b_1 t + b_2$

$$y_e(t) = \frac{\int_0^t y(\lambda)\left(\frac{-(3\lambda^2 - 2t\lambda)}{b_0\sin(b_1\lambda)+b_2} + \frac{(\lambda^3 - t\lambda^2)b_0 b_1\cos(b_1\lambda)}{(b_0\sin(b_1\lambda)+b_2)^2}\right)d\lambda}{t^2} + \frac{\int_0^t y(\lambda)(6\lambda - 2t)d\lambda + \int_0^t \frac{k(t-\lambda)\lambda^2 u(\lambda)}{b_0\sin(b_1\lambda)+b_2}d\lambda}{t^2} \tag{27}$$

Then, one substitutes the expression (26) (or (27)) of $y_e(t)$ in the expression (23) (or (24)) of $y^{(1)}(t)$ such that one obtains $y_e^{(1)}(t)$ as an expression of the estimate of the successive time derivatives of the measured output y.

Step 3: Reconstruction of state.

4a) $\tau(t) = a_0 t + a_1$, one obtains:

$$\begin{cases} \widehat{x}_1 = \dfrac{\int_0^t y(\lambda)\left(\frac{-\left(2a_0\lambda^3+3a_1\lambda^2-t(a_0\lambda^2+2a_1\lambda)\right)}{\tau(\lambda)^2}+(6\lambda-2t)\right)d\lambda+k\int_0^t \frac{(t-\lambda)\lambda^2 u(\lambda)}{\tau(\lambda)}d\lambda}{t^2} \\ \widehat{x}_2 = \dfrac{\int_0^t y(\lambda)\left(\frac{a_0\lambda^2+2a_1\lambda}{\tau(\lambda)^2}-2\right)d\lambda+k\int_0^t \frac{\lambda^2 u(\lambda)}{\tau(\lambda)}d\lambda+2ty_e(t)-\frac{t^2 y_e(t)}{\tau(t)}}{t^2} \end{cases}$$

4b) $\tau(t) = b_0(\sin b_1 t) + b_2$, one obtains:

$$\begin{cases} \widehat{x}_1 = \dfrac{\int_0^t y(\lambda)\left(\frac{-(3\lambda^2-2t\lambda)}{\tau(\lambda)}+\frac{(\lambda^3-t\lambda^2)b_0 b_1\cos(b_1\lambda)}{\tau(\lambda)^2}+6\lambda-2t\right)d\lambda+\int_0^t \frac{k(t-\lambda)\lambda^2 u(\lambda)}{\tau(\lambda)}d\lambda}{t^2} \\ \widehat{x}_2 = \dfrac{\int_0^t y(\lambda)\left(\frac{2\lambda}{\tau(\lambda)}-\frac{\lambda^2 b_0 b_1\cos(b_1\lambda)}{\tau(\lambda)^2}-2\right)d\lambda+\int_0^t \frac{k\lambda^2 u(\lambda)}{\tau(\lambda)}d\lambda+2ty_e(t)-\frac{t^2 y_e(t)}{\tau(t)}}{t^2} \end{cases}$$

B. Simulation

Hereafter, good estimates and robustness with respect to noise are depicted. The initial conditions are: $x_2(0) = 0(rad/s)$, $x_1(0) = 1(rad)$. $k = 1$ and the input voltage is chosen as $u(t) = 12\sin(t)$.

Simulations are given for a polynomial parameter (Fig. 1): $\tau = a_0 t + a_1$ (with $a_0 = 0.001$, $a_1 = 1$) and a sinusoidal one (Fig. 2): $\tau = b_0 \sin(b_1 t) + b_2$ (with $b_0 = 2$, $b_1 = 0.2 * \pi$, $b_2 = 3$).

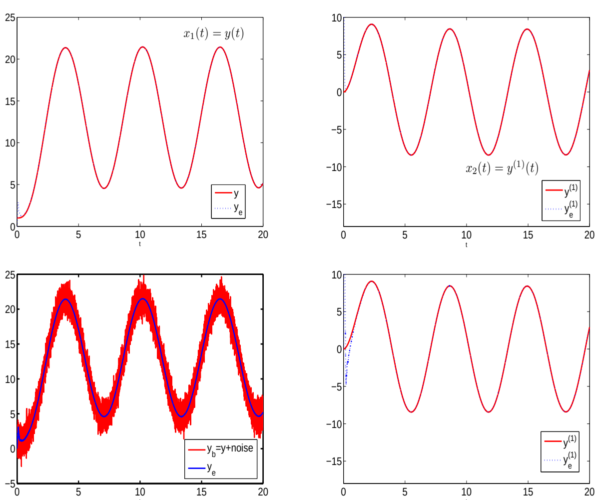

Fig. 1. States and its estimates ($\tau = a_0 t + a_1$).

In the top subfigures, there is no measurement noise. It can be seen that the estimated value tracks quasi-instantaneously exactly the real value. In the bottom subfigures, the measured signal $y(t)$ was perturbed by a white noise uniformly distributed in the interval $[-4, 4]$. It can be seen that this estimator is quite robust w.r.t white noise.

V. CONCLUSION AND PERSPECTIVES

In this paper, an algebraic approach for fast state estimation for linear time-varying systems has been introduced. Additionally, a general exact formula for the estimates has

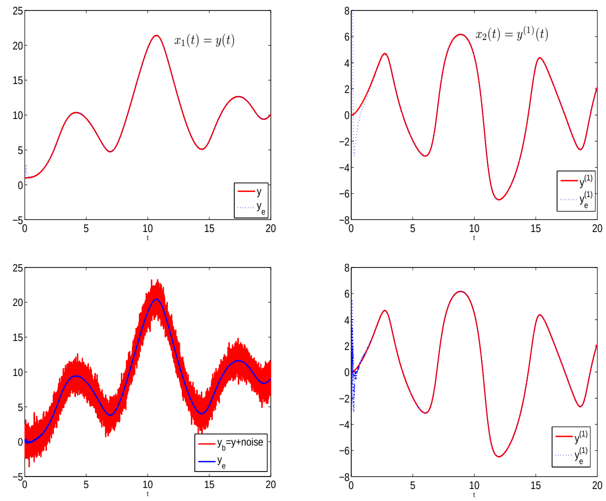

Fig. 2. States and its estimates ($\tau = b_0 \sin(b_1 t) + b_2$).

been derived. Note that the only required condition is that the time-varying parameters have ro be continuously derivable.

REFERENCES

[1] J. P. Barbot, M. Fliess, and T. Floquet. An algebraic framework for the design of nonlinear observers with unknown inputs. *46th IEEE Conference on Decision and Control*, 2007.

[2] E. B. Braiek and F. Rotella. Design of observers for non linear time variant systems. *International Conference on Systems, Man and Cybernetics,*, 4:216 – 220, 1993.

[3] M. Fliess and S. Diop. Nonlinear observability, identifiability and persistent trajectories. *Proc. 36th IEEE Conf. Decision Control*, 1991.

[4] M. Fliess and J. Rudolph. Local tracking observers for flat systems. *Symposium on Control, Optimization and Supervision, CESA '96 IMACS Multiconference*, pages 213–217, 1996.

[5] M. Fliess and J. Rudolph. Corps de hardy et observateurs asymptotiques locaux pour systmes diffrentiellement plats. *C. R. Acad. Sci. Paris Sr. IIb*, 324:513–519, 1997.

[6] M. Fliess and H. Sira-Ramirez. An algebraic framework for linear identification. *ESAIM Control Optim.*, 9:151–168, 2003.

[7] M. Fliess and H. Sira-Ramirez. Closed-loop parametric identification for continuous-time linear systems via new algebraic techniques. In *Continuous-Time Model Identification from Sampled Data*. Springer, 2008.

[8] A. Gensior, J. Weber, H. Guldner, and J. Rudolph. An algebraic parameter identification algorithm and asymptotic observers for estimation of the load of a boost converte. *IEEE international Symposium on Industrial Electronics*, 2007.

[9] R. E. Kalman. A new approach to linear filtering and prediction problems. *Transactions of the ASME - Journal of Basic Engineering*, 82-D:35–45, 1960.

[10] R. E. Kalman and R. S. Bucy. New results in linear filtering and prediction theory. *Transactions of the ASME - Journal of Basic Engineering*, 83:95–107, 1961.

[11] E. Kreindler and P. E. Sarachik. On the concepts of controllability and observability of linear systems. *IEEE Trans. Automat. Control*, 9:129–136, 1964.

[12] D. G. Luenberger. Observers for multivariable systems. *IEEE Trans. Automat. Control*, 11(2):190–197, 1966.

[13] M. Mboup, C. Join, and M. Fliess. A revised look at numerical differentiation with an application to nonlinear feedback control. *15th Mediterranean Conference on Control and Automation*, 2007.

[14] J. O'Reilly. Observers for linear systems. *Academic Press,*, 1983.

[15] L. M. Silverman and H. E. Meadows. Controllability and observability in time-variable linear systems. *SIAM J.Control*, 5:64–73, 1967.

[16] J. Trumpf. Observers for linear time-varying systems. *Linear Algebra Appl*, 425(2-3):303–312, 2007.

[17] K. Yosida. *Operational Calculus*. Springer, New York, 1984.